Chapter 1

INTRODUCTION TO CALCULUS

In the English language, the rules of grammar are used to speak and write effectively. Asking for a cookie at the age of ten was much easier than when you were first learning to speak. These rules developed over time. Calculus developed in a similar way. Sir Isaac Newton and Gottfried Wilhelm von Leibniz independently organized an assortment of ideas and methods that were circulating among the mathematicians of their time. As a tool in the service of science, calculus served its purpose very well. More than two centuries passed, however, before mathematicians had identified and agreed on its underlying principles—its grammar. In this chapter, you will see some of the ideas that were brought together to form the underlying principles of calculus.

CHAPTER EXPECTATIONS

In this chapter, you will

- simplify radical expressions, **Section 1.1**
- use limits to determine the slope and the equation of the tangent to a graph, **Section 1.2**
- pose problems and formulate hypotheses regarding rates of change, **Section 1.3, Career Link**
- calculate and interpret average and instantaneous rates of change and relate these values to slopes of secants and tangents, **Section 1.3**
- understand and evaluate limits using appropriate properties, **Sections 1.4, 1.5**
- examine continuous functions and use limits to explain why a function is discontinuous, **Sections 1.5, 1.6**

Review of Prerequisite Skills

Before beginning this chapter, review the following concepts from previous courses:

- determining the slope of a line: $m = \frac{\Delta y}{\Delta x}$
- determining the equation of a line
- using function notation for substituting into and evaluating functions
- simplifying algebraic expressions
- factoring expressions
- finding the domain of functions
- calculating average rate of change and slopes of secant lines
- estimating instantaneous rate of change and slopes of tangent lines

Exercise

1. Determine the slope of the line passing through each of the following pairs of points:

a. $(2, 5)$ and $(6, -7)$

b. $(3, -4)$ and $(-1, 4)$

c. $(0, 0)$ and $(1, 4)$

d. $(0, 0)$ and $(-1, 4)$

e. $(-2.1, 4.41)$ and $(-2, 4)$

f. $\left(\frac{3}{4}, \frac{1}{4}\right)$ and $\left(\frac{7}{4}, -\frac{1}{4}\right)$

2. Determine the equation of a line for the given information.

a. slope 4, y-intercept -2

b. slope -2, y-intercept 5

c. through $(-1, 6)$ and $(4, 12)$

d. through $(-2, 4)$ and $(-6, 8)$

e. vertical, through $(-3, 5)$

f. horizontal, through $(-3, 5)$

3. Evaluate for $x = 2$.

a. $f(x) = -3x + 5$

b. $f(x) = (4x - 2)(3x - 6)$

c. $f(x) = -3x^2 + 2x - 1$

d. $f(x) = (5x + 2)^2$

4. For $f(x) = \frac{x}{x^2 + 4}$, determine each of the following values:

a. $f(-10)$

b. $f(-3)$

c. $f(0)$

d. $f(10)$

5. Consider the function f given by $f(x) = \begin{cases} \sqrt{3 - x}, & \text{if } x < 0 \\ \sqrt{3 + x}, & \text{if } x \geq 0 \end{cases}$

Calculate each of the following:

a. $f(-33)$

b. $f(0)$

c. $f(78)$

d. $f(3)$

6. A function s is defined for $t > -3$ by $s(t) = \begin{cases} \frac{1}{t}, \text{ if } -3 < t < 0 \\ 5, \text{ if } t = 0 \\ t^3, \text{ if } t > 0 \end{cases}$

Evaluate each of the following:

a. $s(-2)$ b. $s(-1)$ c. $s(0)$ d. $s(1)$ e. $s(100)$

7. Expand, simplify, and write each expression in standard form.

a. $(x - 6)(x + 2)$
b. $(5 - x)(3 + 4x)$
c. $x(5x - 3) - 2x(3x + 2)$
d. $(x - 1)(x + 3) - (2x + 5)(x - 2)$
e. $(a + 2)^3$
f. $(9a - 5)^3$

8. Factor each of the following:

a. $x^3 - x$
b. $x^2 + x - 6$
c. $2x^2 - 7x + 6$
d. $x^3 + 2x^2 + x$
e. $27x^3 - 64$
f. $2x^3 - x^2 - 7x + 6$

9. Determine the domain of each of the following:

a. $y = \sqrt{x + 5}$
b. $y = x^3$
c. $y = \dfrac{3}{x - 1}$
d. $h(x) = \dfrac{x^2 + 4}{x}$
e. $y = \dfrac{6x}{2x^2 - 5x - 3}$
f. $y = \dfrac{(x - 3)(x + 4)}{(x + 2)(x - 1)(x + 5)}$

10. The height of a model rocket in flight can be modelled by the equation $h(t) = -4.9t^2 + 25t + 2$, where h is the height in metres at t seconds. Determine the average rate of change in the model rocket's height with respect to time during

a. the first second
b. the second second

11. Sacha drains the water from a hot tub. The hot tub holds 1600 L of water. It takes 2 h for the water to drain completely. The volume of water in the hot tub is modelled by $V(t) = 1600 - \frac{t^2}{9}$, where V is the volume in litres at t minutes and $0 \le t \le 120$.

a. Determine the average rate of change in volume during the second hour.
b. Estimate the instantaneous rate of change in volume after exactly 60 min.
c. Explain why all estimates of the instantaneous rate of change in volume where $0 \le t \le 120$ result in a negative value.

12. a. Sketch the graph of $f(x) = -2(x - 3)^2 + 4$.
b. Draw a tangent line at the point $(5, f(5))$, and estimate its slope.
c. Estimate the instantaneous rate of change in $f(x)$ when $x = 5$.

CAREER LINK Investigate

CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

Differential calculus is fundamentally about the idea of instantaneous rate of change. A familiar rate of change is heart rate. Elite athletes are keenly interested in the analysis of heart rates. Sporting performance is enhanced when an athlete is able to increase his or her heart rate at a slower pace (that is, to get tired less quickly). A heart rate is described for an instant in time.

Heart rate is the instantaneous rate of change in the total number of heartbeats with respect to time. When nurses and doctors count heartbeats and then divide by the time elapsed, they are not determining the instantaneous rate of change but are calculating the average heart rate over a period of time (usually 10 s). In this chapter, the idea of the derivative will be developed, progressing from the average rate of change calculated over smaller and smaller intervals until a limiting value is reached at the instantaneous rate of change.

Time (s)	Time (min)	Number of Heartbeats
10	0.17	9
20	0.33	19
30	0.50	31
40	0.67	44
50	0.83	59
60	1.00	75

Case Study—Assessing Elite Athlete Performance

The table shows the number of heartbeats of an athlete who is undergoing a cardiovascular fitness test. Complete the discussion questions to determine if this athlete is under his or her maximum desired heart rate of 65 beats per minute at precisely 30 s.

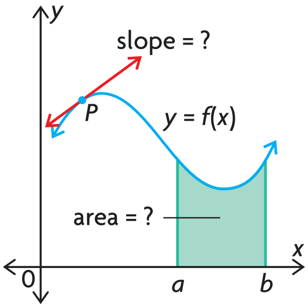
DISCUSSION QUESTIONS

1. Graph the number of heartbeats versus time (in minutes) on graph paper, joining the points to make a smooth curve. Draw a second relationship on the same set of axes, showing the resting heart rate of 50 beats per minute. Use the slopes of the two relationships graphed to explain why the test results indicate that the person must be exercising.
2. Discuss how the average rate of change in the number of heartbeats over an interval of time could be calculated using this graph. Explain your reasoning.
3. Calculate the athlete's average heart rate over the intervals of [0 s, 60 s], [10 s, 50 s], and [20 s, 40 s]. Show the progression of these average heart rate calculations on the graph as a series of secants.
4. Use the progression of these average heart-rate secants to make a graphical prediction of the instantaneous heart rate at $t = 30$ s. Is the athlete's heart rate less than 65 beats per minute at $t = 30$ s? Estimate the heart rate at $t = 60$ s.

What Is Calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

- The problem of tangents: What is the slope of the tangent to the graph of a function at a given point P?
- The problem of areas: What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?



Interest in the problem of tangents and the problem of areas dates back to scientists such as Archimedes of Syracuse (287–212 BCE), who used his vast ingenuity to solve special cases of these problems. Further progress was made in the seventeenth century, most notably by Pierre de Fermat (1601–1665) and Isaac Barrow (1630–1677), a professor of Sir Isaac Newton (1642–1727) at the University of Cambridge, England. Professor Barrow recognized that there was a close connection between the problem of tangents and the problem of areas. However, it took the genius of both Newton and Gottfried Wilhelm von Leibniz (1646–1716) to show the way to handle both problems. Using the analytic geometry of Rene Descartes (1596–1650), Newton and Leibniz showed independently how these two problems could be solved by means of new operations on functions, called differentiation and integration. Their discovery is considered to be one of the major advances in the history of mathematics. Further research by mathematicians from many countries using these operations has created a problem-solving tool of immense power and versatility, which is known as calculus. It is a powerful branch of mathematics, used in applied mathematics, science, engineering, and economics.

We begin our study of calculus by discussing the meaning of a tangent and the related idea of rate of change. This leads us to the study of limits and, at the end of the chapter, to the concept of the derivative of a function.